Interaction energy of Chern-Simons vortices in the gauged O(3) sigma model.

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Abstract

The purpose of this Letter is to present a computation of the interaction energy of gauged O(3) Chern-Simons vortices which are infinitely separated. The results will show the behaviour of the interaction energy as a function of the constant coupling the potential, which measures the relative strength of the matter self-coupling and the electromagnetic coupling. We find that vortices attract each other for $\lambda > 1$ and repel when $\lambda < 1$. When $\lambda = 1$ there is a topological lower bound on the energy. It is possible to saturate the bound if the fields satisfy a set of first order partial differential equations.

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In recent years the study of vortices has become a subject dealing with both particle physics and condensed matter physics. The physics of the recently discovered high-T_c superconductors is an important problem. The studies of Chern-Simons (C-S) solitons can be related to the unusual behaviour of this new type of superconductor.

The purpose of this Letter is to present a computation of the interaction energy of gauged O(3) Chern-Simons vortices. We discuss the solutions analytically and numerically, and present the numerical results. The corresponding problem for vortices of the Abelian Higgs model was first considered by Jacobs and Rebbi [1], as expected, our results are analagous to these. The corresponding problem for Abelian Chern-Simons Higgs vortices was considered in [2].

There is an example in which this crossover, from attractive to repulsive behaviour occurs, namely, the case of Abelian Skyrme CP^1 vortices [3]. In this case no Bogomol'nyi bound occurs, yet crossover takes place.

The essential feature of C-S solitons of all $2+1$ dimensional theories is that the time component of the Euler-Lagrange equation arising from the variation of the gauge connection is used to eliminate the time component of the gauge connection from the stress tensor density in the static field configuration. The energy, which is the integral of this static stress tensor density, is bounded from below by a topological charge and is finite. The static C-S solitons then are the solutions of the static second order equations of this two dimensional subsystem. The problem of gauging sigma models, which was considered sometime ago by Fadde'ev [4] both in $2+1$ and in $3+1$ dimensions, is a pertinent one both physically and for its own sake. Recently there has been renewed interest in this area, thus far in $2+1$ dimensions. First the CP^1 model was gauged with a Chern-Simons $U(1)$ field in Refs.[5], and soon after that the $U(1)$ gauging of the CP^1 model with a Maxwell field, and including a Skyrme term was presented in Ref.[3].

More recently Schroers[8] gauged the $O(3)$ sigma model with a Maxwell $U(1)$ field, much in the same spirit as in the earlier work of Ref.[4]. The O(3) sigma model was gauged with Chern-Simons dynamics in [11]. It is interesting to ask if the model supports attractive and repulsive phases.

1 The $O(3)$ Model

We start by defining the Lagrangian on $2+1$ dimensional Minkowski space,

$$\mathcal{L} = \frac{\kappa}{2\sqrt{2}} \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + (D_\mu \phi^a)^2 - 4V(\phi^3) \tag{1}$$

where the three component field $\phi^a = (\phi^\alpha, \phi^3)$ with $\alpha = 1, 2$, is constrained by $\phi^a \phi^a = 1$, and the covariant derivative $D_\mu \phi^a$ is defined as in [3], and with the opposite sign of the coupling used in [8], as

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A_\mu \varepsilon^{\alpha\beta} \phi^\beta, \qquad\qquad D_\mu \phi^3 = \partial_\mu \phi^3. \tag{2}$$

The Lagrangian (1) is $U(1)$ gauge invariant by virtue of $\mathring{2}$. The potential function, not yet specified, is allowed only to depend only on the $U(1)$ invariant component ϕ^3 of ϕ^a. The energy momentum tensor of this Lagrangian is

$$T_{\mu\nu} = 2 D_\mu \phi^a D_\nu \phi^a - g_{\mu\nu} (D_\mu \phi^a D_\nu \phi^a - 4V) \tag{3}$$

The Hamiltonian density is given by;

$$T_{00} = \mathcal{H} = D_0 \phi^a D_0 \phi^a + D_i \phi^a D_i \phi^a + 4V + (A_0 \phi^a)^2. \tag{4}$$

The essential feature of all Chern-Simons solitons of all $2+1$ dimensional theories is that the time component of the Euler-Lagrange equation arising from the variation of the gauge connection is used to eliminate the time component of the gauge connection from the stress tensor density in the static field configuration. In the static limit this is solved for A_0 yielding;

$$A_0 = -\frac{\kappa}{2\sqrt{2}} \frac{\varepsilon^{ij} F_{ij}}{(\phi^\alpha)^2}. \tag{5}$$

In the static limit $T_{00} = \mathcal{H}$ reduces to;

$$\mathcal{H} \left[\frac{\kappa^2}{2} \frac{F_{ij}^2}{(\phi^\alpha)^2} + (D_i \phi^a)^2 + 4V \right] \tag{6}$$

The choice of potential function will be dictated by the requirement that the volume integral of $\mathcal{H}$ is bounded below by a topological charge. This fixes the potential uniquely. The topological charge density is not a total divergence,

but is only locally a total divergence. The usual winding number density is $\varrho_0 = \varepsilon_{ij}\varepsilon^{abc}\partial_i\phi^a\partial_j\phi^b\phi^c$, which is related to its gauged version

$$\varrho_1 = \varepsilon_{ij}\varepsilon^{abc}D_i\phi^a D_j\phi^b\phi^c.$$

The lower bound on the volume integral of the static Hamiltonian density (6) can be inferred from the following inequalities;

$$(\varepsilon_{ij}D_i\phi^a - \varepsilon^{abc}D_j\phi^b\phi^c)^2 \geq 0, \quad (\frac{\kappa}{\sqrt{2}|\phi^\alpha|}F_{ij} - \sqrt{2}\varepsilon_{ij}V)^2 \geq 0. \tag{7}$$

When the expansion of the static Hamiltonian density is examined, the potential is specified uniquely by the requirement that the energy density be bounded below by a total divergence. The lower bound can be arranged to be equal to the winding number density ϱ_0 plus a total divergence by identifying the potential uniquely as;

$$V = \frac{1}{4\kappa^2}(1-\phi^3)^3(1+\phi^3)$$

The requirement that the surface integral should vanish is satisfied by choosing appropriate boundary conditions. In the case of topologically stable solutions, the asymptotic conditions

$$\lim_{|\vec{x}|\to 0}\phi^3 = -1, \qquad \lim_{|\vec{x}|\to\infty}\phi^3 = \pm 1 \tag{8}$$

guarantee that the volume integral of ϱ yields a nonzero integer winding number. This statement assumes that A_i does not grow too fast at infinity, an assumption which is amply justified as will be seen below when we specialise to the radial field configuration. The conditions (8) with the *upper sign* pertain to the *topologically stable* solutions of nonzero winding, while those with the *lower sign* to the *nontopological* vortices. In the latter case, the winding number vanishes for all vorticities N, in this case the energy is given a lower bound by the magnetic flux.

2 Self Dual Solutions

The topological inequality is saturated when the inequalities (7) are saturated (for the sake of definiteness we have chosen the value of $\kappa = 1$),

yielding the Bogomol'nyi equations,

$$F_{ij} = \mp\varepsilon_{ij}(1-\phi^3)^2(1+\phi^3) \tag{9}$$

$$\varepsilon_{ij}D_i\phi^a = \pm\varepsilon^{abc}D_j\phi^b\phi^c, \tag{10}$$

where the lower/upper signs pertain to anti/self-duality. Our radially symmetric Ansatz for the fields A_i and ϕ^a is

$$A_i = \frac{a(r)-N}{r}\varepsilon_{ij}\hat{x}_j \tag{11}$$

$$\phi^\alpha = \sin f(r)\, n^\alpha,\ \phi^3 = \cos f(r) \tag{12}$$

where $\hat{x}_i = \frac{x_i}{r}$ and $n^\alpha = (\cos N\theta, \sin N\theta)$ are unit vectors, with N defined to be an integer.

The Bogomol'nyi equations (9) and (10) now reduce to the following pair of coupled nonlinear first order ordinary differential equations;

$$\frac{a'}{r} = \pm(1-\cos f)^2(1-\cos f), \qquad\qquad f' = \pm\frac{a\sin f}{r}. \tag{13}$$

The *topological* asymptotic conditions, namely (8) with the upper sign, which were chosen in anticipation of our restriction to the antiselfdual case, now read

$$\lim_{r\to 0} f(r) = \pi, \qquad \lim_{r\to\infty} f(r) = 0 \tag{14}$$

which for the field configuration (11) and (12) imply *vorticity* $= -N$. This is the same as in the usual (ungauged) $O(3)$ model where the radially symmetric antiselfdual vortices satisfy the asymptotic conditions (14) , while the selfdual vortices satisfy instead $\lim_{r\to 0} f(r) = 0;\ \lim_{r\to\infty} f(r) = \pi$.

The asymptotic behaviour of the function $a(r)$ in (11) is of no consequence to the topological stability of the soliton, unlike in the cases of the Higgs models [7], and of the gauged $\mathbb{C}P^1$ models [5] [3]. This is because in the latter systems the topological charge, which is again related to the vorticity, is also proportional to the *magnetic flux*. In the case of the gauged $O(3)$ models, defined in [8] and here, the magnetic flux of the solution is not restricted by the requirement of the stability of the soliton. The only constraint on the large r behaviour of $a(r)$ here is the requirement that the surface integral bounding below the static Hamiltonian density, should vanish. This means

that $a(r)$ should not grow faster than the quantity $(\cos f - 1)$ in that region. Since the magnetic flux is proportional to the quantity$[-a(\infty) + a(0)]$, we shall seek solutions for which both $a(\infty)$ and $a(0)$ are finite, since it is reasonable that the solutions we seek correspond to finite magnetic flux field configurations. As explained above, we shall take $a(0) = N$, but will take $a(\infty) = \alpha$, where α is a nonzero constant whose sign will depend on whether we are considering the topological or the nontopological solutions. Corresponding to the asymptotic conditions (14) for the function $f(r)$, we state the asymptotic conditions on the function $a(r)$ as

$$\lim_{r\to 0} a(r) = N, \qquad \lim_{r\to\infty} a(r) = \alpha. \tag{15}$$

3 Non-Self dual solutions

We will not study further the Bogomol'nyi equations, as they have been amply remarked upon in Refs. [9] [10] [11]. It should be noted that in the latter reference, [11], the proof of existence of the solitons has been demonstrated. The $N = 1$ soliton exists, in contrast to the proof of nonexistence of the $N = 1$ soliton which has been given in [8] in the case of the O(3) sigma model with a Maxwell term describing the curvature.

We will examine the Euler-Lagrange equations arising from the static Hamiltonian. Using the Ansatz the static Hamiltonian reduces to the one dimensional subsystem:

$$L = r\left[\left(\frac{a_r}{r\sin f}\right)^2 + f_r^2 + \left(\frac{a\ \sin f}{r}\right)^2 + \lambda_0(\sin f(1-\cos f))^2\right], \tag{16}$$

defined by

$$\int d^2x\,\mathcal{H} = 2\pi\int dr\; L. \tag{17}$$

When the one dimensional Lagrangian is varied the resulting equations are;

$$\begin{aligned} a(r)\sin^4 f + 2\cot(f) a_r f_r + \frac{a_r}{r} - a_{rr} &= 0 \\ \lambda_0 r(1-\cos f)\sin f(\cos f - \cos^2 f + \sin^2 f) + \frac{a^2\cos f\sin f}{r} &- \\ \frac{\cot f\csc^2 f a_r^2}{r} - f_r - r f_{rr} &= 0, \end{aligned}$$

where $f_r = \frac{df}{dr}$. We first examine the equations (18) in the region $r \ll 1$ region. The solution is;

$$
\begin{aligned}
f(r) &= \pi + f_o r^N + f_1 r^{N+2} + O(r^{N+4}) && (18)\\
a(r) &= N + A_o r^{2N+2} + A_1 r^{2N+4} + O(r^{2N+6}) \text{ where,} && (19)\\
\text{If } N &= 1 \begin{cases} f_1 = -\frac{24A_o^2 + f_o^6 - 6f_o^4\lambda_0}{12 f_o^3}, \\ A_1 = -\frac{8A_o f_o^3 - 3f_o^5 - 48 A_o f_1}{36 f_o}, \end{cases} && (20)\\
\text{If } N &\geq 2 \begin{cases} f_1 = -\frac{A_o^2(1+N)^2 - f_o^4\lambda_0}{(1+N) f_o^3}, \\ A_1 = \frac{2A_o f_1 (1+N)}{(2+N) f_o}, \end{cases} && (21)
\end{aligned}
$$

The constants f_o and A_o are fixed by the asymptotic value of the solution at infinity. They are found by using numerical methods, which require correct values of the fields at the boundary.

Using a power series solution of the Bogomol'nyi equations, it is found that the ratio

$$\frac{A_o}{f_o^2} = \frac{1}{N+1} \tag{22}$$

holds. This gives one free parameter in the analytic series which controls the behaviour of the solutions. It will also serve as a check on the numerically evaluated constants.

Now considering the region $r \gg 1$ and anticpating decaying solutions, we linearise the Euler Lagrange equations about their asymptotic values. That is, the equations are linearised in the functions $F(r)$ about $f = 0$ and $A(r)$ around the asymptotic value $a(r) = \alpha$. The Euler Lagrange equation for $f(r)$ is linearised about its asymptotic value of zero, and found to be;

$$\alpha^2 F - r F_r - r^2 F_{rr} = 0.$$

The solutions to this are ;

$$F = \frac{c_1}{r^\alpha} + c_2 r^\alpha.$$

In order to have finite energy solutions the constant c_2 is chosen as zero. The equation for $a(r)$ is also linearised, and found to be;

$$a_{rr} - r a_r = 0,$$

whose solution is $a = c_3 r^2$ or that a is a constant. The latter option is chosen, in order to restrict attention to finite energy solutions. This constant is chosen to be α. In the case of the analysis of the Bogomol'nyi equations there is found to be a restriction of the value of the constant α. It satisfies the inequality; $\alpha > 1$.

4 Numerical Solutions

We have studied the equations numerically using a shooting method. The integration started in the region $r \ll 1$ using as initial data the power series solutions. The constants f_o and A_o have been found which give the correct behaviour, as $r \gg 1$, of the functions $f(r)$ and $a(r)$. The profiles for the functions $f(r)$ for vorticity $N = 2, 4$ and those for $a(r)$ are given in figures 1,2,4,5, and the respective energy density profiles in figures 3 and 6. In table 3 the total energies of the vortices have been calculated for the approximate solutions. If the Bogomolny'i equations are substituted into the one dimensional system the result is;

$$L_0 = \frac{2 f_r a \sin f}{r} + \frac{2 a_r}{r}(1 - \cos f) \tag{23}$$

The total energy of a topological self dual solution can be written as;

$$\begin{aligned} E_{sd} &= 4\pi \int_0^\infty dr \, \frac{d}{dr} \left(a(1 - \cos f) \right) && (24) \\ &= 8\pi |N| = 2\pi (4|N|). && (25) \end{aligned}$$

This last line is found using the boundary conditions for both functions $a(r)$ and $f(r)$. The energy is independent of the choice of the number α. It is simply enough that $a_r(r \to \infty) = 0$. The figures for the energy density, in both graphs and tables, are given in units of 2π. The figures of the energy density for the value of constant $\lambda_0 = 1$ is the self dual limit, $4|N|$ in our convention. It can be seen that the calculated values are close approximations to the analytic calculation. In figure 7, the values of total energy are plotted. For the $n = 2$ vortex, these figures are multiplied by 2. It can be seen that the $n = 4$ vortex energy is larger than the $n = 2$ energy, for the regime where $\lambda_0 < 1$. Also it can be seen that the $n = 2$ vortex energy is larger than the

$n = 4$ energy, for the regime where $\lambda_0 > 1$. There is a, "cross over", between attraction and repulsion behaviour. The relations;

$$2\mathcal{E}(n = 2, \lambda_0 < 1) < \mathcal{E}(n = 4, \lambda_0 < 1),\ 2\mathcal{E}(n = 2, \lambda_0 > 1) > \mathcal{E}(n = 4, \lambda_0 > 1),$$

can be seen from the data, and in the graph. This states that a vortex with degree 2 is lighter that a vortex with degree 4, so it is energetically favourable that vortices repel. Clearly in the second relation, attraction is favourable. The boundary conditions for the vortices were chosen to be the anti-selfdual configuration. When the vortices are self dual, that is when $\lambda_0 = 1$ the energy of 4 vortices is twice that of 2 vortices. Only at the value of the constant $\lambda_0 = 1$ are the energies per unit vortex number equal. This means that there is no interaction energy between self dual vortices. It can be shown that the diagonal components of the stress tensor vanish in the self dual configuration. The values of the constants A_o and f_o for the self dual solution should be related by equation (22) . It is seen that the numerically calculated numbers are close to their theoretical ratio.

For the nontopological solution, different boundary conditions are used and the result of calculating the energy is $E_{sd} = 4n + 4\alpha$. Although the value of the energy in the nontopological case depends on the surface integral contribution and is, in fact, the magnetic flux, the energy is arbitrary. We have not examined the nontopological solitons, because of this.

When the present work was completed, the work of Ref. [12] was brought to my attention. Our approaches to the problem of the gauged O(3) sigma model are widely different, in particular, in Ref. [12], the examination of varying a different a coupling constant is made. The present work is different, in the sense that it was carried out in the spirit of Ref. [1], where the Higgs coupling constant is varied.

Acknowledgement:

It is a pleasure to thank Prof. D.H. Tchrakian, who suggested the problem, and for continued helpful discussions.

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Constants	$n = 2$	
λ	A_o	F_o
0.8	-7.90372	-5.0000
0.9	-8.09182	-5.0000
1.0	-8.33333	-5.0000
1.1	-8.53843	-5.0000
1.2	-8.65862	-5.0000

Constants	$n = 4$	
λ	A_o	F_o
0.8	-4.68536	-5.00000
0.9	-4.85028	-5.00000
1.0	-5.00000	-5.00000
1.1	-5.13790	-5.00000
1.2	-5.26629	-5.00000

Constants	Energy	
λ	$E(n = 2, \lambda)$	$E(n = 4, \lambda)$
0.8	7.9408	15.9423
0.9	7.9694	15.9695
1.0	8.0000	16.0000
1.1	8.0411	16.0330
1.2	8.0819	16.0680

Figure 1: Profile of the function $f(r)$ for the vortices with $n = 2$ with $\lambda_0 = 1.2, \ldots, 0.8$.
Figure 2: Profile of the function $a(r)$ for the vortices with $n = 2$ with lower values of α corresponding to lower λ_0.
Figure 3: Profile of the energy density for the $n = 2$ vortices where increasing peaks represent increasing energy and increasing λ_0.
Figure 4: Profile of the function $f(r)$ for the vortices with $n = 4$ with $\lambda_0 = 1.2, \ldots, 0.8$.
Figure 5: Profile of the function $a(r)$ for the vortices with $n = 4$ with lower values of α corresponding to lower λ_0.
Figure 6: Profile of the energy density for the $n = 4$ vortices where increasing peaks represent increasing energy and increasing λ_0.
Figure 7: Graph of the energy of two superimposed vortices, $2 \times \mathcal{E}(\lambda, n = 2)$ and $\mathcal{E}(\lambda, n = 4)$, as a function of λ_0.

This figure "fig1-1.png" is available in "png" format from:

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